A Consideration for the Euler Equation in Macroeconomics

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Abstract

This paper considers the problem of the Euler equation in the infinite horizon optimization problem in dynamic macroeconomics.

We show that the assumption of the two-period Euler equation in the infinite horizon problem yields the Euler equation for the one-period optimization problem.

A. The Euler equation in the infinite horizon optimization problem

Consider the simple infinite horizon optimization problem as follows:

$$\max_{\{c_t\}_{t=1}^{\infty}} U(\boldsymbol{c}) = \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) = u(c_1) + \beta u(c_2) + \cdots$$

This problem stresses that total utility $U(c)$ is maximized by all elements of $c = (c_1, c_2, \dots)$ as its control, and the instantaneous utility $u(\cdot)$ satisfies all necessary conditions, especially concavity, differentiability, and increasing $(u'(c_t) \geq 0)$. Here, the constraint is not written explicitly, but satisfies all necessary conditions, especially convexity.

The Euler equation then implies that the difference in total utility $\boldsymbol{U(c)}$ with respect to all its elements $\boldsymbol{c = (c_1, c_2, \dots)}$ equals zero,

$$dU(\boldsymbol{c}) = \frac{\partial U(\boldsymbol{c})}{\partial c_1} dc_1 + \frac{\partial U(\boldsymbol{c})}{\partial c_2} dc_2 + \cdots + \frac{\partial U(\boldsymbol{c})}{\partial c_t} dc_t + \frac{\partial U(\boldsymbol{c})}{\partial c_{t+1}} dc_{t+1} + \cdots = 0,$$

where instantaneous utility is additive and discounted by β, and therefore,

$$\frac{\partial u(c_1)}{\partial c_1} dc_1 + \beta \frac{\partial u(c_2)}{\partial c_2} dc_2 + \cdots + \beta^t \frac{\partial u(c_t)}{\partial c_t} dc_t + \beta^{t+1} \frac{\partial u(c)}{\partial c_{t+1}} dc_{t+1} + \cdots = 0.$$

We then obtain the Euler equation in the infinite horizon optimization problem as follows:

(A) $$\sum_{t=1}^{\infty} \beta^t \frac{\partial u(c_t)}{\partial c_t} dc_t = 0.$$

B. The Euler equation used in dynamic programming

The Euler equation used frequently in dynamic programming (see Acemoglu [1]) can be written as:

(B) $$\frac{\partial u(c_t)}{\partial c_t} dc_t + \beta \frac{\partial u(c_{t+1})}{\partial c_{t+1}} dc_{t+1} = 0, \quad \textbf{for} \quad \forall \boldsymbol{t} \geq \mathbf{1}.$$

C. The problem with dynamic programming caused by the Euler equation (B)

We assume that the Euler equation (B) holds for all t in the model. We then show that this assumption yields the Euler equation for the one-period optimization problem.

The basics of the logic

For example, consider a terminal date of three with the three-period optimization problem. The Euler equation (B) for the terminal date $t = T = 3$ then becomes

$$\beta^3 \frac{\partial u(c_3)}{\partial c_3} dc_3 + \beta^4 \frac{\partial u(c_4)}{\partial c_4} dc_4 = 0.$$

However, $\beta^4 \frac{\partial u(c_4)}{\partial c_4} dc_4$ does not exist because $T = 3$. By considering it as zero by taking a good intuition, the condition becomes

$$\beta^3 \frac{\partial u(c_3)}{\partial c_3} dc_3 = 0.$$

By inserting the terminal condition into the Euler equation (B) for $t = 2$,

$$\beta^2 \frac{\partial u(c_2)}{\partial c_2} dc_2 + \beta^3 \frac{\partial u(c_3)}{\partial c_3} dc_3 = 0,$$

we obtain

$$\beta^2 \frac{\partial u(c_2)}{\partial c_2} dc_2 = 0.$$

By applying the same logic for the Euler equation (B) for $t = 1$,

$$\beta^1 \frac{\partial u(c_1)}{\partial c_1} dc_1 + \beta^2 \frac{\partial u(c_2)}{\partial c_2} dc_2 = 0,$$

we obtain

$$\beta^1 \frac{\partial u(c_1)}{\partial c_1} dc_1 = 0.$$

Therefore, we obtain the condition

(C) $$\frac{\partial u(c_t)}{\partial c_t} dc_t = 0, \qquad \textbf{for} \qquad \forall \boldsymbol{t} \geq \mathbf{1}.$$

This is the condition for the one-period optimization problem.

To infinity

For the above discussion, increase T sequentially by one. We thus recognize that this problem continues forever.

D. Consideration

In this section, we discuss additive consideration after assuming the two-period overlapping generations (OLG) model.

In the two-period OLG model, the physical environment and the reproduction of the population continue forever and each generation optimizes over their two periods of life.

Therefore, we do not encounter the terminal problem considered here.

E. Summary

We considered the problem caused by the Euler equation in the infinite optimization problem.

For dynamic programming and its Euler equation, refer to Ch. 6 (especially Sect. 6) of Acemoglu [1]. See Ikeda [2] for the basics of monetary policy analysis using the OLG model.

F. References

[1] Acemoglu, Daron, Introduction to Modern Economic Growth, Princeton University Press, Princeton, NJ, 2009.

[2] Ikeda, Taro, "An introduction of a simple monetary policy with savings taxation in the overlapping generations model," Kobe University Discussion Paper 1810, 2018.